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WEAK ASYMPTOTIC DECAY VIA A "RELAXED INVARIANCE PRINCIPLE" FOR A WAVE EQUATION WITH NONLINEAR, NONMONOTONE DAMPING

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Dedicated to Jack Hale on the occasion of his 60^{th} birthday

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ABSTRACT

This paper considers the problem of asymptotic decay as $t \to \infty$ of solutions of the wave equation $u_{tt} - \Delta u = -a(x)\beta(u_t, \nabla u)$, $(t,x) \in \mathbb{R}^+ \times \Omega$ (a bounded, open, connected set in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary), $u = 0$ on $\mathbb{R}^+ \times \partial\Omega$. The nonlinear function β is assumed to be globally Lipschitz continuous, $\beta(y) = o(|y|)$ as $|y| \to \infty$, $\beta(0, y_2, \ldots, y_{N+1}) = 0$, $y_1\beta(y_1, \ldots, y_{N+1}) \geq 0$ for all $y \in \mathbb{R}^{N+1}$; β is not assumed to be monotone in y_1. Under additional restrictions on the kernel of β conditions are given which imply $[u, u_t]$ converges to $[0,0]$ weakly in $H = H_0^1(\Omega) \times L^2(\Omega)$ as $t \to \infty$. The work generalizes earlier results of Dafermos [8] and Haraux [15] where strong decay in H as $t \to \infty$ was obtained in the case $\beta(y_1, \ldots, y_{N+1}) = q(y_1)$, q monotone on $\mathbb{R}$.

AMS (MOS) Subject Classifications: 35L70.

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WEAK ASYMPTOTIC DECAY VIA A "RELAXED INVARIANCE PRINCIPLE" FOR A WAVE EQUATION WITH NONLINEAR, NONMONOTONE DAMPING

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0. Introduction

In [8] Dafermos considered the problem of the asymptotic behavior as $t \to \infty$ of solutions to the weakly damped wave equation

$$\begin{aligned} &\Box u + a(x)q(u_t) = 0 \quad \text{on } \mathbb{R}^+ \times \Omega, \\ &u = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \tag{0.1}$$

where $\Box u \equiv u_{tt} - \Delta u$, Ω is an open, connected, bounded set in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary, and $a \in L^\infty(\Omega)$, $a \geq 0$ a.e. in Ω. Dafermos showed that if meas(supp a) > 0 and q is a continuously differentiable strictly increasing function on $\mathbb{R}$, then for any weak solution u of (0.1) with initial data $u(x,0) = u_0(x)$, $u_t(x,0) = v_0(x)$ in $H = H_0^1(\Omega) \times L^2(\Omega)$, the "state" $[u,u_t] \longrightarrow [0,0]$ strongly in H as $t \to \infty$. Subsequently Haraux [15] generalized Dafermos's result to include cases where q is neither strictly increasing nor smooth but where q does possess a maximal monotone graph. The goal of this paper is to remove the hypothesis of monotonicity completely and replace it with the less restrictive assumption that $q: \mathbb{R} \to \mathbb{R}$ has its graph in the first and third quadrants which may touch the horizontal axis either to the left or right of the origin. As we shall see there is a price paid for weakening the hypothesis on q, namely we must assume $a \in C^\infty(\Omega)$, q is continuous, $q(0) = 0$, q satisfies growth and Lipschitz criteria, and most importantly decay is now only shown to be in the weak topology of H.

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The use of the weak topology in showing asymptotic decay is not new [3,4,5,6,8,25]. The basic ideas used here were exposited by J. M. Ball in [3]. However within the context of Ball's paper [3] one would require q viewed as a map $L^2(\Omega) \longrightarrow L^2(\Omega)$ be weakly sequentially continuous. Since this is not the case for general (nonlinear) continuous functions q on $\mathbb{R}$, it is pleasant to report that the results here show that such a weak sequential continuity hypothesis is unnecessary.

In line with the above remarks one may readily note that (0.1) may be written in the first order form (see Section 3)

$$\frac{dU}{dt} = AU + F(U)$$
$$U(0) = U_0 \in H \tag{0.2}$$

where A is the infinitesimal generator of a linear C^0 semigroup e^{At} on a real Hilbert space H and $F: H \longrightarrow H$ is nonlinear, continuous. For such systems various extensions of the useful LaSalle Invariance Principle [16,17] have been given to prove decay to equilibrium. In fact such ideas date back to the work of Hale and Infante [14], Hale [13], and Zubov [30]. More recent results may be found in the papers of Ball [3], Ball and Slemrod [4,5], Brezis [6], Chafee and Infante [7], Dafermos [6], Dafermos and Slemrod [9], Haraux [15], Pazy [21], Webb [28], and books of Hale [13], Haraux [16], Henry [17], Pavel [20], Saperstone [23], and Walker [27]. However in all the work to date (except for the paper of Ball and Slemrod [5]) at least one of the following hypotheses has been made

(i) e^{At} is "smoothing" i.e. e^{At} is compact map: $H \longrightarrow H$ for $t > t_0 \geq 0$. This is useful in "parabolic" like problems (see [3,7,12,13,17,21].)

(ii) $-A-F$ is a maximal monotone operator. This is useful in "hyperbolic" problems when $-F$ itself is monotone (see [6,9,15,16,20,23,27]).

(iii) $\|e^{At}\| \leq Me^{-\alpha t}$ for some $\alpha > 0$. This is useful when A itself generates a strong dissipative mechanism (see [28]).

(iv) $-A$ is monotone and F is weakly sequentially continuous. This useful in "hyperbolic" problems where the nonlinear terms have sufficiently few derivatives (see [3,4]).

Surprising perhaps the simple case when $-A$ is monotone and F is continuous (which includes (0.1)) with no weak continuity or montonicity assumptions on $-F$ is still open (modulo the special cases considered in [5]). It is in this respect that the results given here may shed some light on the general problem of decay to equilibrium of infinite dimensional dynamical systems.

The paper is divided into four sections after this one. Section 1 provides basic definitions and concepts from the theory of nonlinear semigroups. Section 2 shows how semilinear equations of evolution generate nonlinear semigroups. Section 3 then presents details as to how a more general version of (0.1) may be placed in semilinear form. Finally Section 4 derives a "relaxed invariance" principle (Corollary 4.7) which applies to (0.1), (0.2). The main tools used here are Young measures to express composite weak limits as expected values and the concept of generalized evolution equation.

1. Preliminary results on nonlinear semigroups

Definitions. Let H be a real Hilbert space. A (generally nonlinear) semigroup $T(t)$ on H is a family of continuous maps $T(t): H \longrightarrow H$, $T \in \mathbb{R}^+$, satisfying (i) $T(0) =$ identity, (ii) $T(t+s) = T(t)T(s)$, for all $t,s \in \mathbb{R}^+$.

For $\chi \in H$ define the positive orbit through by $\mathcal{O}^+(\chi) = \bigcup_{t \in \mathbb{R}^+} T(t)\chi$. The ω–limit set of χ is the (possibly empty) set $\omega(\chi) = \{\gamma \in H;$ there exists a sequence $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $T(t_n)\chi \longrightarrow \gamma$ as $n \longrightarrow \infty\}$. The weak ω–limit set of χ is the (possibly empty) set given by $\omega_w(\chi) = \{\gamma \in H;$ there exists a sequence $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $T(t_n)\chi \rightharpoonup \gamma$ as $n \longrightarrow \infty\}$. Here $\rightharpoonup$ denotes weak convergence in H.

2. Preliminary results on nonlinear evolution equations

Consider the initial value problem

$$\frac{dU}{dt} = AU + F(U)\,, \tag{2.1}$$

$$U(t_0) = U_0\,, \tag{2.2}$$

where A is the infinitesimal generator of a linear C^0 semigroup e^{At} on a real Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, $F\colon H \longrightarrow H$ is a given function and $U_0 \in H$ is a given initial datum.

Definition [1]. Let $t_1 > t_0$. A function $U \in C([t_0,t_1];H)$ is a weak solution of (2.1), (2.2) on $[t_0,t_1]$ if $U(t_0) = U_0$, $F(U(\cdot)) \in L^1(t_0,t_1;H)$ and if for each $W \in D(A^*)$ the function $(U(t),W)$ is absolutely continuous on $[t_0,t_1]$ and satisfies

$$\frac{d}{dt}(U(t),W) = (U(t),A^*W) + (F(U(t),W)$$

for almost all $t \in [t_0 t_1]$.

Theorem 2.1 [1],[3]. Let $t_1 > t_0$. A function $U\colon [t_0,t_1] \longrightarrow H$ is a weak solution of (2.1), (2.2) on $[t_0,t_1]$ if and only if $F(U(\cdot)) \in L^1(t_0,t_1;H)$ and U satisfies the variation of constants formula

$$U(t) = e^{A(t-t_0)}\,U_0 + \int_{t_0}^{t} e^{A(t-s)}\,F(U(s))ds$$

for all $t \in [t_0,t_1]$.

Theorem 2.2 [22]. Let $F: H \longrightarrow H$ be locally Lipschitz in U. Then for each $U_0 \in H$, (2.1), (2.2) has a unique weak solution U defined on a maximal interval of existence $[t_0, t_{max})$, $t_{max} > t_0$, $U \in C([t_0, t_{max}); H)$. Moreover if $U_n \in C([t_0, t_1]; H)$ are weak solutions of (2.1), (2.2) such that $U_n(0) \longrightarrow U_0$ as $n \longrightarrow \infty$ and $t_1 > t_0$, then $U_n \longrightarrow U$ in $C([t_0, t_1]; H)$ as $n \longrightarrow \infty$, where U is the unique weak solution of (2.1), (2.2) satisfying $U(0) = U_0$. Furthermore for any weak solution U with $t_{max} < \infty$ there holds

$$\lim_{t \uparrow t_{max}} \|U(t)\| = \infty .$$

Theorem 2.3 [4]. Let $F: H \longrightarrow H$ satisfy

(i) F is locally Lipschitz,

(ii) $(F(U), U) \leq 0$ for all $U \in H$.

Then (2.1), (2.2) possesses a unique weak solution $U(t; U_0)$ on $\mathbb{R}^+$ for each $U_0 \in H$. Furthermore $T(t)U_0 = U(t; U_0)$ defines a semigroup on H, $\omega_w(U_0)$ is a nonempty set for each $U_0 \in H$.

The proofs of Theorems 2.1, 2.2, 2.3 may be found in the indicated references. Of course Theorem 2.3 follows directly from Theorem 2.2 since $\|U(t)\| \leq \|U_0\|$ by (ii) and hence $t_{max} = \infty$ and orbits are weakly precompact in H.

3. The damped wave equation

In this section we show how a generalization of the nonlinearly damped wave equation (0.1) may be placed within the semigroup formalism of Section 2.

Let Ω be a bounded, open, connected subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary. Let $\beta: \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be such that

(i) β is globally Lipschitz continuous with Lipschitz constant L

$(|\beta(y)-\beta(z)| \le L|y-z|)$,

(ii) $\beta(0,y_2,\dots,y_{N+1}) = 0$ for all $y \in \mathbb{R}^{N+1}$,

(iii) $\beta(y) = o(|y|)$ as $|y| \longrightarrow \infty$,

(iv) $y_1\beta(y) \ge 0$ for all $y \in \mathbb{R}^{N+1}$.

Let $a \in C^{\infty}(\Omega)$, $a(x) \ge 0$ and $a \not\equiv 0$.

Consider the nonlinearly damped wave equation

$$\Box u = -a(x)\beta(u_t,\nabla u) \quad \text{on } \mathbb{R}^+ \times \Omega \tag{3.1}$$

$$u = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega \tag{3.2}$$

with initial data

$$\left.\begin{array}{l} u(x,0) = u_0(x), \\ u_t(x,0) = v_0(x). \end{array}\right\} \quad x \in \Omega \tag{3.3}$$

If we set $u_t = v$, $U = \begin{bmatrix} u \\ v \end{bmatrix}$ then (3.1), (3.3) may be written in the first order form

$$\frac{dU}{dt} = AU + F(U), \tag{3.4}$$

$$U(0) = U_0, \tag{3.5}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}, \quad F(U) = \begin{bmatrix} 0 \\ -a(x)\beta(v) \end{bmatrix},$$

$$U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

Set $H = H_0^1(\Omega) \times L^2(\Omega)$ where H is endowed with the "energy" inner product

$$(U,\tilde{U}) = (\nabla u,\nabla\tilde{u})_{L^2(\Omega)} + (v,\tilde{v})_{L^2(\Omega)}$$

for

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \text{ in } H .$$

It is well known that A with

$$D(A) = \{[u,v];\ u \in H^2(\Omega) \times H_0^1(\Omega),\ v \in H_0^1(\Omega)\}$$

generates a C^0 group of isometries on H. Furthermore since

$$\begin{aligned}
\|F(U)-F(\tilde{U})\| &= \|a(\cdot)(\beta(v,\nabla u) - \beta(\tilde{v},\nabla\tilde{u}))\|_{L^2(\Omega)} \\
&\le \sqrt{2}\, C_1 \left(\|v-\tilde{v}\|^2_{L^2(\Omega)} + \|\nabla u-\nabla\tilde{u}\|^2_{L^2(\Omega)}\right)^{1/2} \\
&\le \sqrt{2}\, C_1 \|[u,v] - [\tilde{u},\tilde{v}]\| \\
&\le \sqrt{2}\, C_1 \|U-\tilde{U}\|^2, \quad C_1 = (\sup a(x))L,
\end{aligned}$$

we see F is globally Lipschitz.

Also we note

$$(F(U),U) = -(a(\cdot)\beta(v,\nabla u),v)_{L^2(\Omega)} \le 0 .$$

Hence Theorem 2.3 applies and $U(t;U_0) = T(t)U_0$ defines a nonlinear semigroup on $H = H_0^1(\Omega) \times L^2(\Omega)$. To determine the asymptotic behavior of $T(t)U_0$ as $t \longrightarrow \infty$ we will now investigate the properties of the nonempty weak ω–limit set associated with $U_0 \in H$.

4. Asymptotic behavior of solutions to the damped wave equation

As before set $U = [u,v]$, the weak solution (3.1)–(3.3). Then a standard approximation argument [3] shows

$$\|U(t;U_0)\|^2 - \|U_0\|^2 = -2\int_0^t (a(\cdot)\beta(v,\nabla u),v)_{L^2(\Omega)}\, ds \le 0 \tag{4.1}$$

and so

$$\|U(t;U_0)\| \le \|U_0\| \quad \text{for all} \quad t \in \mathbb{R}^+.$$

Now fix $[\varphi,\psi] \in \omega_w(U_0)$, i.e. there exists $t_n \longrightarrow \infty$ so that

$$\left.\begin{aligned} u(t_n;u_0,v_0) &\rightharpoonup \varphi \quad \text{in} \quad H_0^1(\Omega) \\ v(t_n;u_0,v_0) &\rightharpoonup \psi \quad \text{in} \quad L^2(\Omega) \end{aligned}\right\} \quad \text{as } n \longrightarrow \infty.$$

For this sequence t_n consider the translates

$$U_n(t) \overset{\text{def}}{=} U(t+t_n;U_0)\,.$$

Certainly

$$\|U_n(t)\| \le \|U_0\|$$

so that for any fixed $T > 0$

$$\int_0^T \|U_n(t)\|^2 dt \le \|U_0\|^2 T$$

and $\{U_n\}$ is a bounded sequence in $L^2((0,T);H)$.

Lemma 4.1. The translate sequence $U_n = [u_n, v_n]$ possesses a subsequence also denoted by $[u_n, v_n]$ so that

$$U_n = [u_n, v_n] \longrightarrow \overline{U} = [\bar{u}, \overline{w}] \quad \text{in} \quad L^2((0,T);H)$$

and

$$U_n = [u_n, v_n] \longrightarrow \overline{U} = [\bar{u}, \overline{w}] \quad \text{in} \quad C([0,T];H_w) .$$

Here H_w denotes H endowed with weak topology; $[\bar{u}, \overline{w}] \in L^2((0,T);H) \cap C([0,T];H_w)$. In particular

$$\begin{cases} u_n \rightharpoonup \bar{u} & \text{in } L^2((0,T);H_0^1(\Omega)), \\ \nabla u_n \rightharpoonup \nabla \bar{u} & \text{in } L^2(Q_T)^N, \\ v_n \rightharpoonup \bar{v} & \text{in } L^2(Q_T), \end{cases} \tag{4.2}$$

where $Q_T = (0,T) \times \Omega$.

Proof. Weak convergence of a subsequence of U_n in $L^2((0,T);H)$ is obvious. Not quite so obvious is the fact that $[u_n,v_n] \longrightarrow [\bar{u},\bar{w}]$ in $C([0,T];H_w)$ but this follows from the argument of Ball and Slemrod [4, Theorem 2.3].

We now recall a result of M. Schonbek [24] on the representation of weak limits in terms of Young measures (see also Tartar [26], Young [29]).

Proposition 4.2. Let O be an open set in $\mathbb{R}^m$. Let $w_n: O \mapsto \mathbb{R}^q$ be a sequence of functions uniformly bounded in $(L^p(O))^q$ for some $p > 1$. Then there exists a subsequence $\{w_{n_k}\}$ and a family of probability measures $\{\nu_y\}_{y \in O}$ on $\mathbb{R}^q$ so that if $f \in C(\mathbb{R}^q;\mathbb{R})$ and satisfies $f(w) = o(|w|^p)$ as $|w| \longrightarrow \infty$ then

$$f(w_{n_k}) \rightharpoonup \langle \nu_y, f(\lambda)\rangle = \int_{\mathbb{R}^q} f(\lambda)d\nu_y(\lambda)$$

in the sense of distributions.

From the above proposition we can immediately state the following lemma.

Lemma 4.3. For the translates $\{v_n,\nabla u_n\} \subset L^2(Q_T)^{N+1}$ there exists a subsequence $\{v_{n_k},\nabla u_{n_k}\} \subset L^2(Q_T)^{N+1}$ and a family of probability measures $\{\nu_{x,t}\}_{(x,t)\in Q_T}$ on $\mathbb{R}^{N+1}$ such that if $f \in C(\mathbb{R}^{N+1};\mathbb{R})$ and satisfies $f(y) = o(|y|^2)$ as $|y| \longrightarrow \infty$ then

$$f(v_{n_k},\nabla u_{n_k}) \longrightarrow \langle \nu_{x,t}, f(\lambda)\rangle = \int_{\mathbb{R}^{N+1}} f(\lambda)d\nu_{x,t}(\lambda)$$

in the sense of distributions.

Proof. Apply Proposition 4.2 with $O = Q_T$, $m = q = N+1$, $p = 2$, $w_n = \{v_n, \nabla u_n\}$.

For simplicity of notation we write the subsequence $\{v_{n_k}, \nabla u_{n_k}\}$ of Lemma 4.3 as $\{v_n, \nabla u_n\}$.

Lemma 4.4. The following limits hold:

$$\beta(v_n, \nabla u_n) \rightharpoonup \langle \nu_{x,t}, \beta(\lambda)\rangle \quad \text{in} \quad L^2(Q_T);$$

$$v_n \beta(v_n, \nabla u_n) \rightharpoonup \langle \nu_{x,t}, \lambda_1 \beta(\lambda)\rangle \quad \text{in the sense of distributions,}$$

$$\lambda = (\lambda_1, \ldots, \lambda_{N+1});$$

where $\nu_{x,t}$ is the probability measure of Lemma 4.3.

Proof. Since $\beta(y) = o(|y|)$ as $|y| \longrightarrow \infty$ both $\beta(y)$ and $y_1\beta(y)$ are $o(|y|^2)$ as $|y| \longrightarrow \infty$. From Lemma 4.3 we know $\beta(v_n, \nabla u_n) \longrightarrow \langle \nu_{x,t}, \beta(\lambda)\rangle$ and $v_n\beta(v_n, \nabla u_n) \longrightarrow \langle \nu_{x,t}, \lambda_1\beta(\lambda)\rangle$ in the sense of distributions. But since β is Lipschitz,

$$\|\beta(v_n, \nabla u_n)\|_{L^2(Q_T)} \le L\|v_n\|_{L^2(Q_T)} .$$

The uniform boundedness of $\beta(v_n, \nabla u_n)$ and density of $C_0^\infty(Q_T)$ in $L^2(Q_T)$ imply $\beta(v_n, \nabla u_n) \rightharpoonup \langle \nu_{x,t}, \beta(\lambda)\rangle$ in $L^2(Q_T)$.

Lemma 4.5. $\mathcal{U} \in C([0,T]; H_w)$, $\mathcal{U}(0) = [\varphi, \psi]$, and for each $W \in D(A)$ the function $(\mathcal{U}(t), W)$ is absolutely continuous on $[0,T]$ and satisfies the generalized evolution equation

$$\frac{d}{dt}(\overline{U}(t),W) = (\overline{U}(t),A^* w) + \left[\left[\begin{array}{c} 0 \\ a(\cdot)\langle\beta(\lambda),\ \nu_{x,t}(\lambda)\rangle \end{array}\right], W\right] \tag{4.3}$$

for almost all $t \in [0,T]$. (Here $\nu_{x,t}$ is the probability measure of Lemma 4.3.)

Proof. Since U_n are weak solutions of (3.1) – (3.3) we know for all $W \in D(A^*) = D(A)$, $t \in [0,T]$,

$$(U_n(t),W)-(U(t_n),W) = \int_0^t (U_n(s),A^* W)ds + \int_0^t (F(U_n(s)),W)ds\ . \tag{4.4}$$

By Lemma 4.1, $U_n \longrightarrow \overline{U}$ in $C([0,T];H_w)$ so $(U_n(t),W) \longrightarrow (\overline{U}(t),W)$ for $t \in [0,T]$. Moreover the Lebesgue dominated convergence theorem implies

$$\int_0^t (U_n(s),A^* W)ds \longrightarrow \int_0^t (\overline{U}(s),A^* W)ds$$

for $t \in [0,T]$. Also by the definition of $[\varphi,\psi]$ $U(t_n) \rightharpoonup [\varphi,\psi]$ in H. Finally since

$$F(U_n(s)) = \left[\begin{array}{c} 0 \\ a(\cdot)\beta(v_n,\ \nabla u_n) \end{array}\right]$$

and $\beta(v_n,\nabla u_n) \rightharpoonup \langle\beta(\lambda),\nu_{x,t}\rangle$ in $L^2(Q_T)$ (by Lemma 4.4) we see

$$\int_0^t (F(U_n(s)),W)ds \longrightarrow \int_0^t \left[\left[\begin{array}{c} 0 \\ a(\cdot)\langle\beta(\lambda),\ \nu_{x,t}\rangle \end{array}\right], W\right] ds$$

for $t \in [0,T]$. Inserting this limit (4.4) we find that

$$(\overline{U}(t),W)-([\varphi,\psi],W) = \int_0^t (\overline{U}(t),A^*W)ds$$

$$+\int_0^t \left[\begin{bmatrix} 0 \\ a(\cdot)\langle\beta(\lambda),\nu_{x,t}\rangle \end{bmatrix}, W\right]ds\,. \tag{4.5}$$

As the right hand side of (4.5) is absolutely continuous (4.3) follows immediately.

The concept of such a generalized evolution equation involving Young measures was originally suggested by DiPerna [10] within the context of viscosity limits for hyperbolic systems of conservation laws. Within the context of limiting equations on the ω–limit set of an ordinary differential equation, it was Artstein [1] who realized that an ordinary differential equation would not be sufficient to characterize the limit flow of non–autonomous ordinary differential equations. Here we see that similar ideas may be useful in describing the motion of autonomous infinite dimensional dynamical systems on their weak ω–limit sets.

We can now state and prove our main results.

Definition. We set $\ker\beta = \{\lambda \in \mathbb{R}^N;\ \beta(\lambda) = 0\}$.

Theorem 4.6. Let $Q_T' = Q_T \cap \operatorname{supp} a$. Let $\nu_{x,t}$ be the probability measure of Lemma 4.3. Then we have for almost all $x,t \in Q_T'$

$$\operatorname{supp}\nu_{x,t} \subseteq \ker\beta\,. \tag{4.6}$$

Proof. From the "energy" equality (4.1) we know

$$\|U_n(t)\|^2 - \|U(t_n;U_0)\|^2 = -2\int_0^t (v_n, a(\cdot)\beta(v_n,\nabla u_n))_{L^2(\Omega)}\, ds, \tag{4.7}$$

for $t \in [0,T]$. Furthermore the function $\|U(t;U_0)\|$ is nonincreasing and bounded from below so its limit as $t \longrightarrow \infty$ exists. But since $U_n(t) = U(t+t_n;U_0)$ it follows that

$$\lim_{n\to\infty} \|U_n(t)\| - \lim_{n\to\infty} \|U(t_n;U_0)\| = 0$$

and hence by (4.7)

$$\lim_{n\to\infty} \int_0^t (v_n, a(\cdot)\beta(v_n,\nabla u_n))_{L^2(\Omega)}\, ds = 0\,, \tag{4.8}$$

for $t \in [0,T]$.

Now let $\Phi \in C_0^\infty(Q_T)$, $0 \le \Phi \le 1$. Since $v_n a(x)\beta(v_n,\nabla u_n) \ge \Phi v_n a(x)\beta(v_n,\nabla u_n) \ge 0$ for all $x,t \in Q_T$, from (4.8) we find

$$\lim_{n\to\infty} \int_0^T \int_\Omega \Phi(x,t) v_n(x,t) a(x) \beta(v_n,\nabla u_n)\, dxdt = 0\,.$$

But now we use Lemma 4.4 and the fact that $a\Phi \in C_0^\infty(Q_T)$ to conclude

$$\int_0^T \int_\Omega \Phi(x,t) a(x) \langle \lambda_1 \beta(\lambda),\, \nu_{x,t} \rangle\, dxdt = 0 \tag{4.9}$$

where now Φ may be any nonnegative $C_0^\infty(Q_T)$ test function. Equation (4.9) implies $a(x)\langle\lambda_1\beta(\lambda),\nu_{x,t}\rangle = 0$ a.e. in Q_T and so $\langle\lambda_1\beta(\lambda),\nu_{x,t}\rangle = 0$ a.e. in Q_T'. Hence the support of $\nu_{x,t}$ must be contained in the kernel of $\lambda_1\beta(\lambda)$ for almost all $x,t \in Q_T'$. By hypothesis (ii) on β $\ker \lambda_1\beta(\lambda) = \ker \beta(\lambda)$ and the proof is complete.

Corollary 4.7. Let $U_0 \in H$ and $U(t;U_0) = [u,v]$ denote the weak solution of (3.1) – (3.3). Let $[\varphi,\psi]$ denote any arbitrary element of the nonempty weak ω–limit $\omega_w(U_0)$. Then there is a probability measure $\nu_{x,t}$ with $\sup \nu_{x,t} \subseteq \ker \beta$ a.e. in Q_T' and a weak solution $\bar{u}$ of the wave equation

$$\Box\bar{u} = 0 \quad \text{on } \mathbb{R}^+ \times \Omega \tag{4.10}$$

with Dirichlet boundary conditions

$$\bar{u} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega \tag{4.11}$$

satisfying

$$\left.\begin{aligned} \bar{u}(x,0) &= \varphi(x), \\ \bar{u}_t(x,0) &= \psi(x), \end{aligned}\right\} \quad x \in \Omega, \tag{4.12}$$

and the constraints

$$\bar{u}_t(x,t) = \langle\lambda_1,\nu_{x,t}\rangle; \tag{4.13}$$

$$(\nabla\bar{u}(x,t))_i = \langle\lambda_{i+1},\nu_{x,t}\rangle, \quad i = 1,\ldots,N; \tag{4.14}$$

a.e. in Q_T'.

Furthermore if

$$\ker \beta \subseteq \{\lambda \in \mathbb{R}^{N+1};\ c_i \le \lambda_i \le d_i,\ i = 1,\dots,N+1, \text{ for constant } c,d \in \lambda^{N+1}\}$$

then

$$c_1 \le \bar{u}_t(x,t) \le d_1\ ; \tag{4.15}$$

$$c_{i+1} \le (\nabla\bar{u}(x,t))_i \le d_{i+1}\ , \quad i = 1,2,\dots,N; \tag{4.16}$$

a.e. in Q_T'.

Proof. By Lemma 4.3 and 4.5 for any $[\varphi,\psi] \in \omega_w(U_0)$ there is a probability measure $\nu_{x,t}$ and an element $\bar{U} \in C([0,T];H_w)$, $\bar{U}(0) = [\varphi,\psi]$ and $\bar{u}$ is a weak solution to the wave equation with homogeneous Dirichlet boundary conditions on $0 < t < T$. But as $\bar{U} \in L^\infty((0,\infty);H)$ and T is arbitrary we see $\bar{u}$ satisfies the wave equation with Dirichlet boundary conditions on $0 < t < \infty$. The rest of the results are obvious.

Corollary 4.8. Assume $a(x) \ge 0$ with $E = \operatorname{supp} a$, meas $E > 0$. Assume either $\ker \beta \subsetneq \{y \in \mathbb{R};\ y_1 \ge 0\}$ or $\{y \in \mathbb{R}^N;\ y_1 \le 0\}$. Then for any weak solution $U(t;U_0)$ of (3.1) – (3.3) with initial data $U_0 \in H_0^1(\Omega) \times L^2(\Omega)$ we have $U(t;U_0) \to 0$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \longrightarrow \infty$.

Proof. We shall apply Corollary 4.7 and an argument of Haraux [15]. First we note that $\bar{u},\bar{v}$ satisfy $\Box\bar{u} = 0$, $\bar{u}_t = v$, and hence are almost periodic in t, $t \in \mathbb{R}^+$. Also by (4.15) $\bar{u}_t(x,t) \ge 0$

(or ≤ 0) a.e. in Q'_T and since T is arbitrary we note this is true on $Q'_{\mathbb{R}^+}$ as well. Hence the function $\bar{z}(t) = \int_E \bar{u}(x,t)dx$ is monotone in t for $0 < t < \infty$. But because $\bar{u}$ is almost periodic in t, $0 < t < \infty$, $\bar{z}$ must be constant and thus $\bar{u}_t(x,t) = 0$ a.e. in $Q'_{\mathbb{R}^+}$. But now we can adopt the following "principle of C.M. Dafermos" [8,15]:

Let $\tilde{u} \in C(\mathbb{R};H^1_0(\Omega)) \cap C^1(\mathbb{R};L^2(\Omega))$ be a solution of the wave equation $\Box\tilde{u} = 0$ where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is bounded, open, connected with smooth boundary. Assume that for some measurable set $E \subset \Omega$ such that meas $E > 0$ we have $\tilde{u}_t(x,t) = 0$ a.e. on $\mathbb{R}^+ \times E$. Then we conclude: $\tilde{u} = 0$ on $\mathbb{R}^+ \times \Omega$.

Applying this principle to $\bar{u}$ we find $\bar{u} \equiv 0$ and hence $\omega_w(U_0) = \{0\}$.

We note Haraux [15] has shown strong decay in the case $\beta(y_1,\dots,y_{N+1}) = q(y_1)$, q possesses a maximal monotone graph. In the above result the decay is weak but no monotonicity is assumed on β.

We observe for example if $q(\xi)$ satisfies (i) q is globally Lipschitz, (ii) $q(0) = 0$, (iii) $q(\xi) = o(|\xi|)$ as $|\xi| \to \infty$, $\xi q(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ and $\ker q \subseteq \mathbb{R}^+$ or $\ker q \subseteq \mathbb{R}^-$ Corollary 4.8 applies.

The following corollary shows that less restrictive assumptions on β yield decay to a constrained solution of the wave equation.

Corollary 4.9. Let $\ker \beta \subseteq \{y \in \mathbb{R}^{N+1}; c_1 \leq y_1 \leq d_1\}$ and $a(x) > 0$. Then for any weak solution $U(t;U_0)$ of (3.1)–(3.3) with initial data $U_0 \in H^1_0(\Omega) \times L^2(\Omega)$ and a sequence $\{t_n\}$, $t_n \longrightarrow \infty$, there is a subsequence also denoted by $\{t_n\}$ so that

$$[u(t+t_u;u_0,v_0), v(t+t_n;u_0,v_0)] \rightharpoonup [\bar{u}(t;\varphi,\psi),\bar{v}(t;\varphi,\psi)]$$

in $H_0^1(\Omega) \times L^2(\Omega)$ $0 \le t < \infty$, where $\bar{u}$ satisfies (4.10), (4.11), (4.12) and $c_1 \le \bar{v}(x,t) \le d_1$ a.e. in $\Omega \times (0,\infty)$.

Proof. The proof follows directly from Lemma 4.1 and Corollary 4.7.

One final remark. Since the procedure outlined above relies upon first extending the original evolution equation to a generalized evolution equation on the weak ω–limit set, Z. Artstein motivated by ideas in control theory and the calculus of variations has suggested the term "relaxed invariance principle" for the method used here. Hence the origin of the title of this paper.

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